Optimal Quantum Feedback Control for Canonical Observables

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Abstract

We show that the stochastic Schrödinger equation for the filtered state of a system, with linear free dynamics, undergoing continual non-demolition measurement or either position or momentum, or both together, can be solved explicitly within a class of Gaussian states which we call extended coherent states. The asymptotic limit yields a class of relaxed states which we describe explicitly. Bellman's principle is then applied directly to optimal feedback control of such dynamical systems and the Hamilton Jacobi Bellman equation for the minimum cost is derived. The situation of quadratic performance criteria is treated as the important special case and solved exactly for the class of relaxed states.

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1   Introduction

Quantum noise was originally developed to model irreversible quantum dynamical systems, where it played an external and secondary role, however, the realization that it could be measured and the results used to influence the system evolution has had a profound effect on its physical status [1],[2],[3]. The great leap forward since then has been made by experimentalists who have made the practical implementation of quantum state estimation and adaptive feedback control a reality. With this, has come new problems that have received intense interest in the physics community [4-10].

In this paper, we wish to treat the problem of how to describe the quantum evolution of a system with linear free dynamics when we perform non-demolition measurements of, typically both, canonical position and momentum. The problem where position measurements only are made has been of historical importance. In this situation, the model is the one considered by Ghirardi, Rimini and Weber [11], who also obtained the asymptotic form for the state. The

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asymptotic solution, with explicit reference to the stochastic Schrödinger equation within the Itô formulation, was first given by Diósi [12], see also Belavkin and Staszewski [13]. Essentially, the solution to the stochastic Schrödinger equation could be understood as an randomly parameterized Gaussian state. The parameters being mean position, mean momentum and a complex inverse variance. We shall show that the same class of states, which we term extended coherent states, suffice for the stochastic Schrödinger equation describing simultaneous monitoring of position and momentum.

The problem of optimal quantum feedback control can then be tackled at this point. Bellman equations have been derived previously for the optimal cost of controlling a qubit system [9]. In fact, the general problem can be understood as a classical control problem on the space of quantum states [14] if one exploits the separation of quantum estimation component from the control component: here we may construct a, typically infinite dimensional, Hamilton Jacobi Bellman theory and are then faced with the problem of finding a sufficient parameterization of states for particular situation. In the case of non-demolition position and momentum measurements, we have that the extended coherent states offer a sufficient parameterization. The quadratic performance problem is the important special case and has been treated by Doherty and Jacobs [15] for feedback from measuring one quadrature of a Bosonic mode. We show that this problem is solvable when both canonical observables are measured.

1.1 Stochastic Schrödinger Equation

Consider a quantum system evolving with free Hamiltonian H while undergoing continual diffusive interaction with several independent apparatuses, each coupling to the system in a Markovian manner with coupling operator L_j for the j-th apparatus. (The $\{L_j\}$ do not generally need to be either commuting or self-adjoint.) The state, ψ_t, of the system continually updated using the output of the apparatuses, will then satisfy a stochastic Schrödinger equation of the type [16],[3],[10],

$$
\begin{aligned}
|d\psi_t\rangle &= \frac{1}{i\hbar} H |\psi_t\rangle \, dt - \frac{1}{2} \sum_j \left(L_j^\dagger L_j - 2\lambda_j(t) L_j + \lambda_j^2(t) \right) |\psi_t\rangle \, dt \\
&\quad + \sum_j \left(L_j - \lambda_j(t) \right) |\psi_t\rangle \, dW_t^{(j)}.
\end{aligned}
\tag{1}
$$

where $\lambda_j(t) = \operatorname{Re} \langle \psi_t | L_j \, \psi_t \rangle$ and $\{W^{(j)}\}$ is a multi-dimensional Wiener process with $dW_t^{(j)} dW_t^{(k)} = \delta_{jk} dt$. This equation was first postulated in the context of filtering by Belavkin where the apparatuses are separate Bose fields and the $W_t^{(j)}$ are innovations processes obtained by de-trending the output processes.

The stochastic Schrödinger equation for measurement of canonically conjugate observables, $\hat{q}$ and $\hat{p}$, has been derived from first principles by Scott and Milburn [17]. They considered a discrete time model with simultaneous measurement of position and momentum by separate apparatuses, and considered

the continuous time limit of progressively more imprecise and frequent measurements. Taking $L_1 = \sqrt{\frac{\kappa}{2}}\hat{q}$ and $L_2 = \sqrt{\frac{\tilde{\kappa}}{2}}\hat{p}$ and denoting the innovations by $W_t^{(1)} = W_t$ and $W_t^{(2)} = \tilde{W}_t$, their particular stochastic Schrödinger equation reads as

$$
\begin{aligned}
|d\psi_t\rangle &= \left(\frac{1}{i\hbar}H - \frac{\kappa}{4}\left(\hat{q} - \langle\hat{q}\rangle_t\right)^2 - \frac{\tilde{\kappa}}{4}\left(\hat{p} - \langle\hat{p}\rangle_t\right)^2\right)|\psi_t\rangle\, dt \\
&\quad + \sqrt{\frac{\kappa}{2}}\left(\hat{q} - \langle\hat{q}\rangle_t\right)|\psi_t\rangle\, dW_t + \sqrt{\frac{\tilde{\kappa}}{2}}\left(\hat{p} - \langle\hat{p}\rangle_t\right)|\psi_t\rangle\, d\tilde{W}_t.
\end{aligned}
\tag{2}
$$

The equation involves the expectations $\langle\hat{q}\rangle_t = \langle\psi_t|\hat{q}\,\psi_t\rangle$ and $\langle\hat{p}\rangle_t = \langle\psi_t|\hat{p}\,\psi_t\rangle$ and is therefore non-linear in the state ψ_t. Here the constants κ m^{-2}s^{-1} and $\tilde{\kappa}$ N^{-2}s^{-3} are positive and describe the measurement strength for the two apparatuses. In general, κ and $\tilde{\kappa}$ has units of inverse variance of position, respectively momentum, per unit time. In [19], the limiting procedure was revisited and, as an alternative to increasingly imprecise measurements, one could use increasingly weak interaction between the apparatuses and the system. The scaling between the imprecision of measurement, or weakness of interaction with the apparatus, and the rate at which the discrete measurements is made must be such as to allow a general central limit effect to take place. In principle, it is possible, to set up the apparatuses to obtain desired values of κ and $\tilde{\kappa}$.

The purpose of [17] was to consider nonlinear dynamics, however, we shall only deal with quadratic Hamiltonians of the type $H = H\left(f, v\right)$

$$
H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}\hbar\mu\hat{q}^2 - f\hat{q} + v\hat{p}.
\tag{3}
$$

Here f and v are external fields which will later be replaced with control functions. We shall show that it is possible to find a general solution for the stochastic state ψ_t, with initial condition being that we start in a coherent state, realized as a random wave function taking values in a special class of wave functions, termed extended coherent states.

2 Extended Coherent States

Let $L^2\left(\mathbb{R}\right)$ be the Hilbert space of square integrable functions of position coordinate x with standard Schrödinger representation of the canonical observables $\hat{q}$ and $\hat{p}$. By an *extended coherent state*, we mean a wave function $\psi\left(\bar{q}, \bar{p}, \eta\right)$, parameterized by real numbers $\bar{q}, \bar{p}$ and a complex number $\eta = \eta' + i\eta''$ where $\eta' > 0$, taking the form

$$
\langle x|\psi\left(\bar{q}, \bar{p}, \eta\right)\rangle = \left(\frac{\eta'}{2\pi}\right)^{1/4}\exp\left\{-\frac{\eta}{4}\left(x - \bar{q}\right)^2 + i\frac{\bar{p}}{\hbar}x\right\}.
\tag{4}
$$

When η is real $(\eta'' = 0)$, the vectors are just the well-known coherent states [18] The distribution of the canonical variables in extended coherent state $\psi\left(\bar{q}, \bar{p}, \eta\right)$

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is Gaussian with characteristic function

$$\langle \exp\{ir\hat{q}+is\hat{p}\}\rangle_{\bar{q},\bar{p},\eta} = \exp\left\{ir\bar{q}+is\bar{p}-\frac{1}{2}\left(C_{qq}r^2+2C_{qp}rs+C_{pp}s^2\right)\right\}, \quad (5)$$

where

$$C_{qq}=\frac{1}{\eta'}, \; C_{qp}=-\frac{\hbar\eta''}{2\eta'}, \; C_{pp}=\frac{\hbar^2}{4}\left(\eta'+\frac{\eta''^2}{\eta'}\right). \quad (6)$$

The mean values of the position and the momentum in an extended coherent state are evidently $\langle\hat{q}\rangle=\bar{q}$ and $\langle\hat{p}\rangle=p$ respectively. We have that C_{qq} is the variance of $\hat{q}$, C_{pp} is the variance of $\hat{p}$, while $C_{qp}=\frac{1}{2}\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle-\langle\hat{p}\rangle\langle\hat{q}\rangle$ is the covariance of $\hat{q}$ and $\hat{p}$.

2.1 Derivation of the Characteristic Function

To establish (5), let us first recall that coherent states may be constructed from creation/annihilation operators $a^{\pm}=\frac{1}{2}\sqrt{\eta'}\hat{q}\pm\frac{1}{i\hbar\sqrt{\eta'}}\hat{p}$ by identifying $\psi\left(\bar{q},\bar{p},\eta'\right)$ as the eigenstate of a^- with eigenvalue $\alpha=\frac{1}{2}\sqrt{\eta'}\bar{q}-\frac{1}{i\hbar\sqrt{\eta'}}\bar{p}$. In particular, if Ω denotes the zero-eigenstate of a^- then

$$\psi\left(\bar{q},\bar{p},\eta'\right)=D_\alpha\,\Omega$$

where $D_\alpha=\exp\{\alpha a^+-\alpha^* a^-\}$ is a Weyl displacement unitary. Next observe that we may obtain extended coherent states from coherent states by the simple application of a unitary transformation:

$$\psi\left(\bar{q},\bar{p},\eta'+i\eta''\right)\equiv V\,\psi\left(\bar{q},\bar{p},\eta'\right)$$

with $V=\exp\left\{-\frac{i}{4}\eta''\left(\hat{q}-\bar{q}\right)^2\right\}$. (This transformation is, in fact, linear canonical.) We may introduce new canonical variables $\hat{q}'$ and $\hat{p}'$ by $\hat{q}'=V^\dagger\hat{q}V\equiv\hat{q}$ and $\hat{p}'=V^\dagger\hat{p}V=\hat{p}-\frac{1}{2}\hbar\eta''\left(\hat{q}-\bar{q}\right)$. We note that $\exp\{ir\hat{q}+is\hat{p}\}=D_z$ where $z=-\frac{1}{2}\hbar\sqrt{\eta'}s+i\frac{1}{\sqrt{\eta'}}r$ and

$$V^\dagger D_z V=\exp\left\{ir\hat{q}'+is\hat{p}'\right\}=D_w\,e^{\frac{1}{2}i\hbar\eta''\bar{q}s}$$

where $w=-\frac{1}{2}\hbar\sqrt{\eta'}s+i\frac{1}{\sqrt{\eta'}}\left(r-\frac{1}{2}\hbar\eta''s\right)$. Using well-known properties for Weyl displacement operators [18] and their Ω-state averages, we find

$$\begin{aligned}
\langle\exp\{ir\hat{q}+is\hat{p}\}\rangle_{\bar{q},\bar{p},\eta} &= \langle\Omega|D_\alpha^\dagger V^\dagger D_z V D_\alpha\,\Omega\rangle\\
&= \langle\Omega|D_\alpha^\dagger D_w D_\alpha\,\Omega\rangle e^{\frac{1}{2}i\hbar\eta''\bar{q}s}\\
&= e^{w\alpha^*-w^*\alpha-\frac{1}{2}|w|^2}e^{\frac{1}{2}i\hbar\eta''\bar{q}s}
\end{aligned}$$

and substituting in for α and w gives the required result.

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2.2 Weyl Independence

We say that the canonical variables are Weyl independent for a given state $\langle \cdot \rangle$, not necessarily pure, if we have the following factorization

$$\langle \exp\{ir\hat{q} + is\hat{p}\} \rangle = \langle \exp\{ir\hat{q}\} \rangle \langle \exp\{is\hat{p}\} \rangle$$

for all real r and s. If the state possesses moments to all orders, then Weyl independence means that symmetrically (Weyl) ordered moments factor according to $\langle : f(\hat{q}) g(\hat{p}) : \rangle = \langle f(\hat{q}) \rangle \langle g(\hat{p}) \rangle$, for all polynomials f, g. By inspection, we see that coherent states leave the canonical variables Gaussian and Weyl-independent. However, the $\eta'' \neq 0$ extended states do not have this Weyl-independence property.

3 Stochastic Wave Function

We now return to the equation (2) for the conditioned state ψ_t. Let $\langle X \rangle_t = \langle \psi_t | X | \psi_t \rangle$, for a general operator X, then we have the following stochastic Ehrenfest equation

$$d\langle X \rangle = \left\{ \frac{1}{i\hbar} \langle [X, H] \rangle - \frac{\kappa}{4} \langle [[X, \hat{q}], \hat{q}] \rangle - \frac{\tilde{\kappa}}{4} \langle [[X, \hat{p}], \hat{p}] \rangle \right\} dt$$

$$+ \sqrt{\frac{\kappa}{2}} \left(\langle X\hat{q} + \hat{q}X \rangle - \langle \hat{q} \rangle \langle X \rangle \right) dW_t + \sqrt{\frac{\tilde{\kappa}}{2}} \left(\langle X\hat{p} + \hat{p}X \rangle - \langle \hat{p} \rangle \langle X \rangle \right) d\tilde{W}_t. \quad (7)$$

For $X = \hat{q}, \hat{p}$, we find

$$d\langle \hat{q} \rangle = \left(\frac{1}{m} \langle \hat{p} \rangle + v \right) dt + \sqrt{2\kappa} C(\hat{q}, \hat{q}) dW_t + \sqrt{2\tilde{\kappa}} C(\hat{q}, \hat{p}) d\tilde{W}_t,$$

$$d\langle \hat{p} \rangle = (-\hbar\mu \langle \hat{q} \rangle + f) dt + \sqrt{2\kappa} C(\hat{q}, \hat{p}) dW_t + \sqrt{2\tilde{\kappa}} C(\hat{p}, \hat{p}) d\tilde{W}_t. \quad (8)$$

where $C(\hat{q}, \hat{q}) = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2$, $C(\hat{p}, \hat{p}) = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$, and $C(\hat{q}, \hat{p}) = \frac{1}{2} \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle - \langle \hat{p} \rangle \langle \hat{q} \rangle$. In the following, we wish to investigate the dynamical evolution of the random state ψ starting from an initial coherent state. It turns out however that we do not remain within the class of coherent states: if we did, then $\hat{q}$ and $\hat{p}$ would remain Weyl-independent and, in particular, $C(\hat{q}, \hat{p})$ would vanish, along with the noise term in the $\langle \hat{p} \rangle$-equation of (8) above and this would lead to an inconsistent system of equations. Fortunately, it turns out that it is possible to think of ψ as evolving as a random state taking values amongst the extended coherent states. Explicitly, we make the ansatz that the state ψ_t takes the form

$$\psi_t = \psi(\bar{q}_t, \bar{p}_t, \eta_t) \quad (9)$$

where $\bar{q}_t$ and $\bar{p}_t$ are real-valued diffusion processes satisfying and η_t is a complex-valued deterministic function. Our assumption that we start from a coherent

state is equivalent to asking that $\eta(0) = \sigma^{-2} > 0$, with σ having the interpretation as the initial dispersion in position.

We shall now show that $\bar{q}$, $\bar{p}$ satisfy the diffusion equations (8), while η satisfies the Riccati equation

$$\frac{d}{dt}\eta = 2\kappa + i2\mu - \frac{1}{2}\left(\tilde{\kappa}\hbar^2 + i\frac{\hbar}{m}\right)\eta^2. \tag{10}$$

3.1 Consistency with the Statistical Evolution

Let r, s be fixed real parameters and set $D = \exp\{ir\hat{q} + is\hat{p}\}$. We shall investigate the evolution through the characteristic function

$$G_t = \langle\psi_t| D |\psi_t\rangle = \langle D\rangle_t.$$

Observing that $[D, \hat{q}] = \hbar s D$, $[D, \hat{p}] = -\hbar r D$ we find

$$dG = \left\{\frac{ir}{2m}\langle\hat{p}D + D\hat{p}\rangle - \frac{is\hbar\mu}{2}\langle\hat{q}D + D\hat{q}\rangle + \left(ifs + ivr - \frac{\hbar^2\left(\kappa s^2 + \tilde{\kappa}r^2\right)}{4}\right)G\right\} dt$$

$$+ \sqrt{\frac{\kappa}{2}}\left(\langle D\hat{q} + \hat{q}D\rangle - \langle\hat{q}\rangle G\right) dW + \sqrt{\frac{\tilde{\kappa}}{2}}\left(\langle D\hat{p} + \hat{p}D\rangle - \langle\hat{p}\rangle G\right) d\tilde{W}.$$

The identity $e^{ir\hat{q}+is\hat{p}} = e^{\frac{1}{2}irs\hbar}e^{ir\hat{q}}e^{is\hat{p}} = e^{-\frac{1}{2}irs\hbar}e^{is\hat{p}}e^{ir\hat{q}}$ (Baker Campbell Hausdorff formula) then allows us to compute that

$$\langle\hat{q}D\rangle = e^{\frac{1}{2}irs\hbar}\frac{1}{i}\frac{\partial}{\partial r}\left(e^{-\frac{1}{2}irs\hbar}G\right) = \left(\bar{q} + i\left(C_{qq}^2 r + C_{qp}^2 s\right) + \frac{1}{2}s\hbar\right)G,$$

and likewise

$$\langle D\hat{q}\rangle = \left(\bar{q} + i\left(C_{qq}r + C_{qp}s\right) - \frac{1}{2}s\hbar\right)G,$$

$$\langle\hat{p}D\rangle = \left(\bar{p} + i\left(C_{qp}r + C_{pp}s\right) + \frac{1}{2}r\hbar\right)G,$$

$$\langle D\hat{p}\rangle = \left(\bar{p} + i\left(C_{qp}r + C_{pp}s\right) - \frac{1}{2}r\hbar\right)G.$$

Hence

$$dG = \frac{ir}{m}\left\{\bar{p} + i\left(C_{qp}r + C_{pp}s\right)\right\}G\,dt - is\hbar\mu\left\{\bar{q} + i\left(C_{qq}r + C_{qp}s\right)\right\}G\,dt$$

$$+ \left(ifs + ivr - \frac{\kappa\hbar^2 s^2}{4} - \frac{\tilde{\kappa}\hbar^2 r^2}{4}\right)G\,dt$$

$$+ i\sqrt{2\kappa}\left(C_{qq}r + C_{qp}s\right)G\,dW + i\sqrt{2\tilde{\kappa}}\left(C_{qp}r + C_{pp}s\right)G\,d\tilde{W}. \tag{11}$$

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Under our ansatz (9), we should also have, by the Itô rule,

$$
\begin{aligned}
dG &= \frac{\partial G}{\partial \bar{q}} d\bar{q} + \frac{\partial G}{\partial \bar{p}} d\bar{p} + \frac{1}{2} \frac{\partial^2 G}{\partial \bar{q}^2} (d\bar{q})^2 + \frac{\partial^2 G}{\partial \bar{q} \partial \bar{p}} (d\bar{q} d\bar{p}) + \frac{1}{2} \frac{\partial^2 G}{\partial \bar{p}^2} (d\bar{p})^2 \\
&\quad + \frac{\partial G}{\partial \eta'} d\eta' + \frac{\partial G}{\partial \eta''} d\eta'' \\
&= irG d\bar{q} + isG d\bar{p} - \frac{1}{2} r^2 G (d\bar{q})^2 - rsG (d\bar{q} d\bar{p}) - \frac{1}{2} s^2 G (d\bar{p})^2 \\
&\quad + \left(\frac{1}{2\eta'^2} r^2 - \frac{\hbar \eta''}{2\eta'} rs - \frac{\hbar^2}{8} \left(1 - \frac{\eta''^2}{\eta'^2} \right) s^2 \right) d\eta' \\
&\quad + \left(\frac{\hbar}{2\eta'} rs - \frac{1}{4} \frac{\hbar^2 \eta''}{\eta'} s^2 \right) d\eta''.
\end{aligned}
\tag{12}
$$

Equating the coefficients of (11) and (12) gives the system of equations

$$
\begin{aligned}
r &: \quad d\bar{q} = \left(\frac{1}{m} \bar{p} + v \right) dt + \sqrt{2\kappa} C_{qq} \, dW + \sqrt{2\tilde{\kappa}} C_{qp} \, d\tilde{W}, \\
s &: \quad d\bar{p} = (-\hbar\mu\bar{q} + f) \, dt + \sqrt{2\kappa} C_{qp} \, dW + \sqrt{2\tilde{\kappa}} C_{pp} \, d\tilde{W}, \\
r^2 &: \quad (d\bar{q})^2 - \frac{1}{\eta'^2} d\eta' = \frac{1}{m} C_{qp} dt + \frac{\tilde{\kappa}\hbar^2}{2} \, dt, \\
s^2 &: \quad (d\bar{p})^2 + \frac{\hbar^2}{4} \left(1 - \frac{\eta''^2}{\eta'^2} \right) d\eta' + \frac{1}{2} \frac{\hbar^2 \eta''}{\eta'} d\eta'' = -2\hbar\mu C_{qp} \, dt + \frac{\kappa\hbar^2}{2} \, dt, \\
rs &: \quad (d\bar{q} d\bar{p}) + \frac{\hbar \eta''}{2\eta'^2} d\eta' - \frac{\hbar}{2\eta'} d\eta'' = \frac{1}{m} C_{pp} \, dt - \hbar\mu \, C_{qq} dt.
\end{aligned}
$$

The first two of these agree exactly with (8), while the next three are entirely consistent with the pair of real equations

$$
\begin{cases}
\dfrac{d}{dt}\eta' = 2\kappa + \dfrac{\hbar}{m}\eta'\eta'' - \frac{1}{2}\tilde{\kappa}\hbar^2 \left(\eta'^2 - \eta''^2 \right), \\[2mm]
\dfrac{d}{dt}\eta'' = 2\mu - \dfrac{\hbar}{2m} \left(\eta'^2 - \eta''^2 \right) - \tilde{\kappa}\hbar^2 \eta'\eta''.
\end{cases}
\tag{13}
$$

Together, they are equivalent to the single complex Riccati equation (10).

3.2   Asymptotic States

The Riccati equation (10) is to be solved in the half plane $\eta' > 0$ of physical solutions and has the unique, globally attractive, fixed point

$$
\eta_\infty = \frac{2}{\hbar} \sqrt[+]{\frac{\kappa + i\mu}{\tilde{\kappa} + \dfrac{i}{m\hbar}}}.
\tag{14}
$$

(Here $\sqrt[+]{\cdot}$ denotes the complex root having positive real part.)

In the case of a harmonic oscillator of frequency ω, we have $\mu = \dfrac{m\omega^2}{\hbar} > 0$ and we may achieve a coherent state (η_∞ real) as the limit state if we tune the measurement strengths such that $\kappa \equiv m^2\omega^2\,\tilde{\kappa}$. In this case, $\eta_\infty \equiv \dfrac{2m\omega}{\hbar}$, corresponding to a coherent state with position uncertainty $\sigma_\infty = \sqrt{\dfrac{\hbar}{2m\omega}}$. Otherwise the limit state will be an extended coherent state.

We should remark that $\sqrt{\dfrac{\kappa}{\tilde{\kappa}}}$ corresponds to the squeezing parameter s introduced in [17] to describe the bias in favor of the $\hat{q}$ or $\hat{p}$ coupling.

4   Optimal Quantum Feedback Control

We fix a terminal time $T > 0$ and let $\{f_t : 0 < t < T\}$ and $\{v_t : 0 < t < T\}$ be prescribed functions which we refer to as control policies. Let $\psi_t = \psi\left(\bar{q}_t, \bar{p}_t, \eta_t\right)$ be the solution to the stochastic Schrödinger equation with time-dependent free Hamiltonian $H = H\left(f_t, v_t\right)$ and initial state being an extended state $\psi\left(\bar{q}_0, \bar{p}_0, \eta_0\right)$ at time t_0 somewhere in the time interval $[0, T]$.

We wish to grade the control policies $\{f_t\}$ and $\{v_t\}$ over the time interval $[t_0, T]$ and do so by assigning a cost $J = J\left[\{f_t\}, \{v_t\}; t_0, T; \bar{q}_0, \bar{p}_0, \eta_0\right]$ taking the general form

$$J\left[\{f_t\}, \{v_t\}; t_0, T; \bar{q}_0, \bar{p}_0, \eta_0\right] = \int_{t_0}^{T} \ell\left(s; f_s, v_s; \bar{q}_s, \bar{p}_s, \eta_s\right) ds + g\left(\bar{q}_T, \bar{p}_T, \eta_T\right).$$
(15)

Here ℓ is a function of time, the current control policy values, and current state parameters. The function g, known as a target or bequest function in control theory, is a function of the state parameters at termination. We assume that both are continuous in their arguments.

The cost J will vary from one experimental trial to another, and must be thought of as a random variable depending on the measurement output. The aim of this section is to evaluate the minimum average cost over all possible control policies, which we denote as

$$S\left(t_0, T; \bar{q}_0, \bar{p}_0, \eta_0\right) = \min_{\{f_t\}, \{v_t\}} \mathbb{E}\left\{J\left[\{f_t\}, \{v_t\}; t_0, T; \bar{q}_0, \bar{p}_0, \eta_0\right]\right\}.$$

4.1   Bellman Optimality Principle

For simplicity, let us write $z \equiv (\bar{q}, \bar{p}, \eta)$ and $u = (f, v)$ and $S \equiv S(t_0; z_{t_0})$, etc. Taking $t_0 < t_0 + \Delta t < T$, we have that

$$S\left(t_0; z_{t_0}\right) = \min_{\{f_t\}, \{v_t\}} \mathbb{E}\left\{\int_{t_0}^{t_0+\Delta t} \ell\left(s; u_s; z_s\right) ds + J\left[\{u_t\}; t_0 + \Delta t, T; z_0 + \Delta z\right]\right\}$$

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where $\Delta z = z_t - z_{t_0}$ is, of course the random change in the state parameters from time t_0 to $t_0 + \Delta t$. We have that

$$\int_{t_0}^{t_0+\Delta t} \ell\left(s; u_s; z_s\right) ds = \ell\left(t_0, u_{t_0}, z_{t_0}\right) \Delta t + o\left(\Delta t\right)$$

up to terms that are small of order in Δt. Likewise, assuming that S will be sufficiently differentiable,

$$
\begin{aligned}
& S\left(t_0 + \Delta t; z_0 + \Delta z\right) \\
=\ & S\left(t_0; z_0\right) + \left.\frac{\partial S}{\partial t}\right|_0 \Delta t + \left.\frac{\partial S}{\partial z}\right|_0 \Delta z + \frac{1}{2}\Delta z' \left.\frac{\partial^2 S}{\partial z^2}\right|_0 \Delta z + o\left(\Delta t\right) \\
=\ & S\left(t_0; z_0\right) + \left.\frac{\partial S}{\partial t}\right|_0 \Delta t + \left.\frac{\partial S}{\partial \bar{q}}\right|_0 \left(\frac{1}{m}\bar{p} + v_t\right)\Delta t + \left.\frac{\partial S}{\partial \bar{p}}\right|_0 \left(-\hbar\mu\bar{q} + f_t\right)\Delta t \\
& + \left.\frac{\partial S}{\partial \eta'}\right|_0 \frac{d\eta'}{dt}\Delta t + \left.\frac{\partial S}{\partial \eta''}\right|_0 \frac{d\eta''}{dt}\Delta t \\
& + \frac{1}{2}\left.\frac{\partial^2 S}{\partial \bar{q}^2}\right|_0 \left[2\kappa C_{qq}^2 + 2\tilde{\kappa}C_{qp}^2\right]\Delta t + \frac{1}{2}\left.\frac{\partial^2 S}{\partial \bar{p}^2}\right|_0 \left[2\kappa C_{qp} + 2\tilde{\kappa}C_{pp}\right]\Delta t \\
& + \left.\frac{\partial^2 S}{\partial \bar{q}\partial \bar{p}}\right|_0 2\sqrt{\kappa\tilde{\kappa}}\left[C_{qq}^2 + C_{pp}^2\right]C_{qp}\Delta t + o\left(\Delta t\right).
\end{aligned}
$$

(On the right hand side, we are evaluating at $t_0, \bar{q}_0, \bar{p}_0, \eta_0$.)

The Bellman principle of optimality [20], see also [21] for instance, states that if $\{u_t^*\}$ is an optimal control policy exercised over the time interval $[t_0, T]$ for a given start state at time t_0, then if we operated this policy up to time $t_0 + \Delta t$ then the remaining component of the policy will be optimal for the control problem over $[t_0 + \Delta t, T]$ with start state being the current (random) state at time $t_0 + \Delta T$. If we assume the existence of such an optimal policy, then, within the above approximations, as $\Delta t \to 0^+$, we are lead to the partial differential equation (Hamilton Jacobi Bellman equation, or just Bellman equation) for $S = S\left(t; \bar{q}, \bar{p}, \eta\right)$

$$
0 = \frac{\partial S}{\partial t} + \mathcal{H}\left(t; \bar{q}, \bar{p}, \eta; \frac{\partial S}{\partial \bar{q}}, \frac{\partial S}{\partial \bar{p}}\right) + \frac{\partial S}{\partial \eta'}\frac{d\eta'}{dt} + \frac{\partial S}{\partial \eta''}\frac{d\eta''}{dt}
$$
$$
+ \frac{\partial^2 S}{\partial \bar{q}^2}\left[\kappa C_{qq}^2 + \tilde{\kappa}C_{qp}^2\right] + 2\frac{\partial^2 S}{\partial \bar{q}\partial \bar{p}}\sqrt{\kappa\tilde{\kappa}}\left[C_{qq} + C_{pp}\right]C_{qp} + \frac{\partial^2 S}{\partial \bar{p}^2}\left[\kappa C_{qp}^2 + \tilde{\kappa}C_{pp}^2\right]
$$
$$(16)$$

where we introduce

$$
\mathcal{H}\left(t; \bar{q}, \bar{p}, \eta; y_q, y_p\right) := \min_{f,v}\left\{y_q\left(\frac{1}{m}\bar{p} + v\right) + y_p\left(-\hbar\mu\bar{q} + f\right) + \ell\left(t; f, v; \bar{q}, \bar{p}, \eta\right)\right\}.
$$

It should perhaps be stressed that the derivation of this equation is entirely classical. The key feature of the Bellman equation is that the minimum is now

taken pointwise: that is we look for the optimal scalar values f, v at a single instant of time. The equation is to be solved subject to the terminal condition $\lim_{t \to T^-} S(t, \bar{q}, \bar{p}, \eta) = g(\bar{q}, \bar{p}, \eta)$.

In principle, once a minimizing solution $f^* = f(t; \bar{q}, \bar{p}, \eta)$, $v^* = v^*(t; \bar{q}, \bar{p}, \eta)$ is known, it may be used as a Markov control for closed loop feedback: that is, the control policies are taken as these functions of the current state parameters.

The Bellman equations arising in quantum feedback control have so far proved to be highly nonlinear and prohibitively hard to solve as a rule. Our equation (16) is no exception, however, the nonlinearities are in due to the η variable. We remark that if we assume that we start off in a state relaxed at the equilibrium value $\eta = \eta_\infty$, then the coefficients of the η', η'' derivatives vanish exactly, and we may take the covariances C_{qq}, C_{qp} and C_{pp} at their relaxed value determined from (6) evaluated at the asymptotic value η_∞. As the relaxation time is typically small, we may justify this for large times T in comparison. This ignores any η-transient contribution to the cost, but at least opens up the possibility of solving the Bellman equation and finding optimal Markov control policies. We give the fundamental class of interest, quadratic performance criteria, next.

4.2   Linear Quantum Stochastic Regulator

We consider the following quadratic control problem not involving any costs on the η parameter. In particular, we make the assumption that the starting state is an asymptotic state ($\eta = \eta_\infty$) and so we ignore η as a variable. We set $x = (\bar{q}, \bar{p})$ and $u = (f, v)$ and take the specific choices

$$\ell(t, u, x) = \frac{1}{2}x' A_t x + \frac{1}{2}u' E_t u,$$
$$g(x) = \frac{1}{2}x' R x,$$

where A_t, E_t and R are 2×2 symmetric matrices with E_t being invertible. The free Heisenberg equations are linear and can be written as $\dot{x}_t = F_t x_t + M_t u$. The control problem is now essentially the same as the classical stochastic regulator [21]. In this case we introduce a dual variable y to x and obtain the $\mathcal{H}$-function

$$\mathcal{H}(t, x, y) = \min_u \{\ell(t, u, x) + y'(F_t x + M_t u)\}$$
$$= \frac{1}{2}x' A_t x + y' F_t x + \min_u \left\{\frac{1}{2}u' E_t u + y' M_t u\right\}$$

with the minimum attained at

$$u^* = -E_t^{-1} M_t' y$$

and we find

$$\mathcal{H}(t, x, y) = \frac{1}{2}x' A_t x + y' F_t x - \frac{1}{2}y' M_t E_t^{-1} M_t' y$$

Seeking an η-independent solution, the Bellman equation (16) reduces to

$$0 = \frac{\partial S}{\partial t} + \mathcal{H}(t, x, \nabla S) + \frac{1}{2} K_{ij} \frac{\partial^2 S}{\partial x_i \partial x_j}.$$

Here K is the matrix of the second order coefficients in (16) and these will be determined by the covariances (6) determined at the asymptotic value η_∞. As is well known [21], the solution takes the form $S(t, x) = \frac{1}{2} x' \Sigma_t x + a_t$ where Σ_t satisfies the matrix Riccati equation

$$\frac{d\Sigma_t}{dt} = -\Sigma_t F_t - F_t' \Sigma_t + \Sigma_t M_t E_t^{-1} M_t' \Sigma_t - A_t, \qquad \Sigma_T = R,$$

while a_t satisfies

$$\frac{da_t}{dt} = -tr\{K\Sigma_t\}, \qquad a_T = 0.$$

The optimal control policy is therefore given by

$$u^*(t, x) = -E_t^{-1} M_t' \nabla S = E_t^{-1} M_t' \Sigma_t x.$$

4.3 Commentary

The sufficiency property of the extended coherent states means that the results above are of importance to the corresponding filtering problem. Indeed this allows us to implement a quantum analogue of the Kalman filter for state estimation amongst the class of extended coherent states. The Kalman filter is of considerable conceptual and practical importance in classical control theory and plays a crucial role in optimal feedback control. In fact, the matrix Riccati equation occurring in linear stochastic regulator also appears in a dual formulation as a Kalman filtering problem [21]. Unfortunately, the solution to the fully parameterized Bellman equation, that is, when we do not start from the equilibrium value $\eta = \eta_\infty$, seems to be disappointingly difficult even in the linear regulator example as the matrix K will be quartic in η. (Such difficulties seem to be sadly the norm in applications to optimal quantum control as a whole, so far.) The control problem is however tractable for the class of relaxed coherent states and corresponds to the linear regulator model for quadratic performance and this at least gives us some insight into possible applications.

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